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A note on the smallest semicopula-based universal integral and an application

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Abstract

In this paper, we study two properties of the seminormed fuzzy integral. By applying these results, we propose alternative proof of the monotone convergence theorems for smallest semicopula-based universal integrals, which are proposed by J. Borzová-Molnárová et al. in 2015.

Keywords: Continuity; Monotone convergence theorem; Semicopula; Smallest semicopula-based universal integral; Strict level-set

1. Introduction

It is known that Sugeno integral by M. Sugeno in [11] is one of the effective tools to model multi-criteria decision problems. It is defined by

$$(S)\int f\,d\mu = \sup_{0\leqslant\alpha} \min\{\alpha, \mu(f_\alpha)\},$$

where $(X, \mathscr{A}, \mu)$ is a continuous monotone measure space, $f : X \to [0, \infty)$ is $\mathscr{A}$-measurable and f_α is the α- level set of f.
Some applications of Sugeno integral are given by [1,9,10].

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• Motivated by its applications, many types of generalized Sugeno integrals have been studied. Of which seminormed fuzzy integral was introduced [3,4,7,8,13] and given by

$$\int_{\mathsf{S}} f d\mu = \mathbf{I}_{\mathsf{S}}(\mu, f) = \sup_{0 \leqslant \alpha \leqslant 1} \mathsf{S}(\alpha, \mu(f_\alpha)),$$

where $(X, \mathscr{A}, \mu)$ is a monotone measure space, S is a semicopula and $f : X \to [0, 1]$ is $\mathscr{A}$-measurable.
In [14], one remarkable property of Sugeno integral stated in Theorem 9.1 is that

$$(S) \int f d\mu = \sup_{0 \leqslant \alpha} \min\{\alpha, \mu(f_{\alpha^+})\}, \tag{1}$$

for every $\mu \in \mathscr{M}^1_{(X,\mathscr{A})}$ and $f \in \mathscr{F}_{(X,\mathscr{A})}$. Where $f_{\alpha^+} = \{x \in X | f(x) > \alpha\}$ the strict α-level set of f.
• Note that in general this result is incorrect for seminormed fuzzy integral (See Example 2.3). Thus, a question is naturally posed that
1) Under what conditions the property (1) holds true for seminormed integral?
• In the process of studying the seminormed fuzzy integral, another question is also raised that
2) Under what conditions the function $\alpha \mapsto \mathsf{S}(\alpha, \mu(f_\alpha))$ reaches its supremum value on $[0, 1]$ i.e., there exists $\beta \in [0, 1]$ such that $\mathbf{I}_{\mathsf{S}}(\mu, f) = \mathsf{S}(\beta, \mu(f_\beta))$.
• The main results of this paper are to provide answers to the above two questions. Namely, we will prove that
1) The equality (1) holds whenever the semicopula S is left-continuous in the first variable.
2) For each $f \in \mathscr{F}_{(X,\mathscr{A})}$ the function $\alpha \mapsto \mathsf{S}(\alpha, \mu(f_\alpha))$ reaches its supremum value on the interval $[0, 1]$ whenever the measure μ is continuous from above and semicopula S is right-continuous in each variable.
• By applying the obtained results we show alternative proofs of Theorems 2.1 and 2.2 in [6].

The layout of the paper is organized as follows: Section 2 provide necessary background. The main results of this paper are shown in Section 3. An application of the main results is given in Section 4. Some versions of monotone convergence theorems are presented in Section 5. Section 6 is a conclusion and finally, Appendix is presented in Section 7.

2. Preliminaries

In this section, we recall and introduce some necessary concepts for next sections.

2.1. Semicopula

Definition 2.1. An operator $\mathsf{S} : [0, 1]^2 \to [0, 1]$ is called a semicopula if it satisfies

1. S is nondecreasing.
2. $\mathsf{S}(1, x) = \mathsf{S}(x, 1) = x$ for all $x \in [0, 1]$.

The following operators are common semicopulas:

$$\mathsf{M}(x, y) = x \wedge y,$$
$$\Pi(x, y) = x \cdot y,$$
$$\mathsf{W}(x, y) = (x + y - 1) \vee 0,$$
$$\mathsf{D}(x, y) = \begin{cases} x \wedge y, & \text{if } x \vee y = 1, \\ 0, & \text{otherwise.} \end{cases}$$

We denote by $\mathfrak{S}$ the class of all semicopulas.

Several concepts of continuity of a semicopula are given by the following definition.

Definition 2.2. Let $\mathsf{S} \in \mathfrak{S}$. Then we say that

1. S is continuous (left-continuous, right-continuous, respectively) in the first variable if the function $x \mapsto \mathsf{S}(x, b)$ is continuous (left-continuous, right-continuous, respectively) on $[0, 1]$ for every $b \in [0, 1]$.
2. S is continuous (left-continuous, right-continuous, respectively) in the second variable if the function $y \mapsto \mathsf{S}(a, y)$ is continuous (left-continuous, right-continuous, respectively) on $[0, 1]$ for every $a \in [0, 1]$.
3. S is right-continuous if for every $(a, b) \in [0, 1]^2$, for every sequence $\{x_n\}, \{y_n\} \subset [0, 1]$ with $x_n \to a^+$, $y_n \to b^+$, it holds $\mathsf{S}(x_n, y_n) \to \mathsf{S}(a, b)$.
4. S is left-continuous if for every $(a, b) \in [0, 1]^2$, for every sequence $\{x_n\}, \{y_n\} \subset [0, 1]$ with $x_n \to a^-$, $y_n \to b^-$, it holds $\mathsf{S}(x_n, y_n) \to \mathsf{S}(a, b)$.

It is easy to see that the continuity (the right-continuity, the left-continuity, respectively) implies the continuity (the right-continuity, the left-continuity, respectively) in each variable.
In general, a real function of two variable may be continuous in each variable without being continuous. The same conclusion for the left-continuity and right-continuity of a two variable function. However, for a semicopula (or a monotone two variable-function in general) we have the following special properties:

Proposition 2.1. (See Proposition 1.19 in [2].) *Let $\mathsf{S} \in \mathfrak{S}$. S is continuous if and only if it is continuous in each variable.*

Corollary 2.1. S *is continuous if and only if* S *is left-continuous and right-continuous.*

Imitating the above proposition, we get the following extended result.

Proposition 2.2. *Let $\mathsf{S} \in \mathfrak{S}$. S is right-continuous (or left-continuous, respectively) if and only if it is right-continuous (or left-continuous, respectively) in each variable.*

Proof. It is given in Appendix. □

Remark 2.1. Note that in practice, checking the continuity (right-continuity, left-continuity, respectively) of a semicopula S is often more difficult than checking the continuity (right-continuity, left-continuity, respectively) of S in each variable. Therefore, Propositions 2.1 and 2.2 are very useful in practice. On the other hand, due to Proposition 2.2, the assumption of the left-continuous and right-continuous of semicopula S in Theorems 1.1 and 1.2 in [6] can be replaced by the left-continuity and right-continuity in each variable, respectively.

To clarify the above definitions, let us study the following examples:

Example 2.1. D is not left-continuous because $\lim_{n\to\infty} \mathsf{D}\left(\frac{1}{2}, 1 - \frac{1}{n}\right) = 0 \neq \frac{1}{2} = \mathsf{D}\left(\frac{1}{2}, 1\right)$.

Example 2.2. Let $\mathsf{S} : [0, 1]^2 \to [0, 1]$ defined by

$$\mathsf{S}(x, y) = \begin{cases} 0, & \text{if } (x, y) \in \left[0, \frac{1}{2}\right) \times [0, 1), \\ \min\{x, y\}, & \text{otherwise.} \end{cases}$$

It is not difficult to verify that $\mathsf{S} \in \mathfrak{S}$. Further, S is right-continuous in the first and right-continuous in the second variable. But it is neither left-continuous in the first variable nor left-continuous in the second variable. Therefore, it is right-continuous but not left-continuous.

2.2. Upper and Lower semicontinuity

In this subsection, we recall the upper and lower semicontinuity and their properties.

Definition 2.3. Let X be a topological space, we say that a function $f : X \to \mathbb{R}$ is upper (or lower) semicontinuous if for every $x \in X$, for every $\varepsilon > 0$ there exists a neighborhood $V_\varepsilon(x)$, such that $t \in V_\varepsilon(x)$ implies $f(t) < f(x) + \varepsilon$ (or $f(x) - \varepsilon < f(t)$).

For the semicontinuity, we have the following fundamental results.

Proposition 2.3. *Let X be a metric space. The following assertions are equivalent:*

1. *$f : X \to \mathbb{R}$ is upper (or lower) semicontinuous.*
2. *For every $\alpha \in \mathbb{R}$, $\{x \in X | f(x) \geqslant \alpha\}$ is a closed set in X (or $\{x \in X | f(x) \leqslant \alpha\}$ is closed).*
3. *For every sequence $\{x_n\} \subset X$ with $\lim_{n\to\infty} x_n = x \in X$, there holds*

$$\limsup_{n\to\infty} f(x_n) \leqslant f(x) \ \ (or \ \liminf_{n\to\infty} f(x_n) \geqslant f(x)).$$

Proposition 2.4. *Let X be a compact topological space. If $f : X \to \mathbb{R}$ is upper (or lower) semicontinuous then f achieves maximum (or minimum) on X.*

Proof. Put $M = \sup\limits_{x\in X} f(x)$. There exists a sequence $\{x_n\} \subset X$ such that

$$\lim_{n\to\infty} f(x_n) = M.$$

It follows from the compactness of X that there exists a subsequence $\left\{x_{n_k}\right\}$ of $\{x_n\}$ such that $x_{n_k} \to x^* \in X$. By applying Proposition 2.3-3, we get

$$M = \lim_{k\to\infty} f\left(x_{n_k}\right) \leqslant \limsup_{k\to\infty} f\left(x_{n_k}\right) \leqslant f\left(x^*\right) \leqslant M.$$

So, f achieves maximum on X. The case of the lower semicontinuity is similar. The proof is finished. □

2.3. Fuzzy measure

Let $(X, \mathscr{A})$ be a measurable space, where $\mathscr{A}$ is a σ-algebra of subsets of nonempty set X.

Definition 2.4. ([14]) Let $\mu : \mathscr{A} \to [0, \infty]$ be a non-negative, extended real-valued set function. Then we say μ is a monotone measure if it satisfies
1. $\mu(\emptyset) = 0$ (vanishing at $\emptyset$).
2. $A, B \in \mathscr{A}$ and $A \subset B$ imply $\mu(A)) \leqslant \mu(B)$ (monotonicity).
The triplet $(X, \mathscr{A}, \mu)$ is called a monotone measure space.
Further, if the monotone measure μ satisfies
3. $\{A_n\} \subset \mathscr{A}$, $A_1 \subset A_2 \subset ...$, and $\bigcup\limits_{n=1}^{\infty} A_n \in \mathscr{A}$ imply $\lim\limits_{n\to\infty} \mu(A_n) = \mu\left(\bigcup\limits_{n=1}^{\infty} A_n\right)$ (continuity from below).
4. $\{A_n\} \subset \mathscr{A}$, $A_1 \supset A_2 \supset ...$, $\mu(A_1) < \infty$, and $\bigcap\limits_{n=1}^{\infty} A_n \in \mathscr{A}$ imply $\lim\limits_{n\to\infty} \mu(A_n) = \mu\left(\bigcap\limits_{n=1}^{\infty} A_n\right)$ (continuity from above).
Then μ is called a continuous monotone measure and the triplet $(X, \mathscr{A}, \mu)$ is called a continuous monotone measure space.

Let $f : X \to [0, \infty]$, we denote by $f_\alpha = \{x \in X | f(x) \geqslant \alpha\}$ the α-level set of f for $\alpha \geqslant 0$, and $f_{\alpha^+} = \{x \in X | f(x) > \alpha\}$ the strict α-level set of f for $\alpha \geqslant 0$.

Definition 2.5. Let $\mathscr{S}$ denote the class of all measurable spaces $(X, \mathscr{A})$.

1. We denote by $\mathscr{F}_{(X,\mathscr{A})}$ the set of all $\mathscr{A}$-measurable functions $f : X \to [0, \infty]$.

2. We denote by $\mathscr{F}^{[0,a]}_{(X,\mathscr{A})}$) the set of all $\mathscr{A}$-measurable functions $f : X \to [0, a]$ for some $a \in (0, \infty]$.
3. For each $b \in (0, \infty]$, we denote by $\mathscr{M}^{b}_{(X,\mathscr{A})}$ the set of all monotone measures satisfying $\mu(X) = b$.

Throughout this paper, the main object of our interest is the class of smallest semicopula-based universal integrals with respect to semicopula S given by:

$$\mathbf{I}_{\mathsf{S}}(\mu, f) = \sup_{0<\alpha} \mathsf{S}(\alpha, \mu(f_\alpha)),$$

where $(X, \mathscr{A}) \in \mathscr{S}$, $(\mu, f) \in \mathscr{M}^{1}_{(X,\mathscr{A})} \times \mathscr{F}_{(X,\mathscr{A})}$ and S is a semicopula.
Note that:
• This integral is also called a S-semicopula integral or a seminormed fuzzy integral e.g. [7,8,13].
• In particular, for $\mathsf{S} = \mathsf{M}$ we recover the original definition of the Sugeno integral [11]. For $\mathsf{S} = \Pi$ the integral $\mathbf{I}_\Pi$ is the Shilkret integral [12].
• Also, for some $A \in \mathscr{A}$ we get

$$\mathbf{I}_{\mathsf{S},A}(\mu, f) := \mathbf{I}_{\mathsf{S}}(\mu, f.\chi_A) = \sup_{0<\alpha} \mathsf{S}\left(\alpha, \mu\left((f.\chi_A)_\alpha\right)\right) = \sup_{0<\alpha} \mathsf{S}(\alpha, \mu(A \cap f_\alpha)).$$

• The property (1) is incorrect for seminormed fuzzy integral in general. Indeed, we consider the following example:

Example 2.3. Let $X = [0, 1]$ and μ be Lebesgue measure on X. Let S be as in Example 2.2 and $f(x) = \frac{1}{2}\chi_{\left[0,\frac{1}{2}\right]}(x)$ on X. Then

$$\mu(f_\alpha) = \begin{cases} \frac{1}{2}, & \text{if } \alpha \in \left[0, \frac{1}{2}\right], \\ 0, & \text{if } \alpha \in \left(\frac{1}{2}, 1\right], \end{cases}$$

and

$$\mu(f_{\alpha^+}) = \begin{cases} \frac{1}{2}, & \text{if } \alpha \in \left[0, \frac{1}{2}\right), \\ 0, & \text{if } \alpha \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

It follows that

$$\mathsf{S}(\alpha, \mu(f_\alpha)) = \begin{cases} \frac{1}{2}, & \text{if } \alpha = \frac{1}{2}, \\ 0, & \text{if } \alpha \neq \frac{1}{2}, \end{cases}$$

and $\mathsf{S}(\alpha, \mu(f_{\alpha^+})) = 0$ for all $\alpha \in [0, 1]$. This implies that

$$\mathbf{I}_{\mathsf{S}}(\mu, f) = \frac{1}{2} \neq 0 = \sup_{\alpha\in[0,1]} \mathsf{S}(\alpha, \mu(f_{\alpha^+})).$$

2.4. The monotone convergence theorems

In 2015, J. Borzová-Molnárová et al. studied monotone convergence theorems for the smallest universal integral in [6]. The authors have stated a very complete form as follows

Theorem 2.1. (Theorem 2.1 in [6]) *Let $\mathsf{S} \in \mathfrak{S}$ be left-continuous and $\mu \in \mathscr{M}^{1}_{(X,\mathscr{A})}$. Then the following assertions are equivalent:*

1. *μ is continuous from below.*
2. *For all $f, f_n \in \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ such that $f_n \nearrow f$ and $f_n \to f$, it holds $\lim_{n\to\infty} \mathbf{I}_{\mathsf{S}}(\mu, f_n) = \mathbf{I}_{\mathsf{S}}(\mu, f)$.*

Theorem 2.2. (Theorem 2.2 in [6]) *Let $\mathsf{S} \in \mathfrak{S}$ be right-continuous and $\mu \in \mathscr{M}^{1}_{(X,\mathscr{A})}$. Then the following assertions are equivalent:*

1. *μ is continuous from above.*
2. *For all $f, f_n \in \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ such that $f_n \searrow f$ and $f_n \to f$, it holds $\lim_{n\to\infty} \mathbf{I}_{\mathrm{S}}(\mu, f_n) = \mathbf{I}_{\mathrm{S}}(\mu, f)$.*

Unfortunately, the proof of theorems are incorrect. Until 2019, in [5] the same authors have pointed out the mistakes and proposed correct proofs. The main idea of the new proof for Theorem 2.1 is to use the representation of $\mathbf{I}_{\mathrm{S}}(\mu, f)$ via simple functions

$$\mathbf{I}_{\mathrm{S}}(\mu, f) = \sup\left\{\mathbf{I}_{\mathrm{S}}(\mu, \varphi) \Big/ \varphi \in \mathscr{F}^{Sim([0,1])}_{(X,\mathscr{A})}, h_{\mu,\varphi} \leqslant h_{\mu,f}\right\}.$$

The main idea for Theorem 2.2 relies on some duality arguments.

• In Section 4, the alternative proofs of Theorems 2.1 and 2.2 will be shown by applying the main results of this paper.

3. Main result

This section is devoted to present some interesting properties of smallest semicopula-based universal integral. At first we need the following two lemmas.

Lemma 3.1. *Let $(X, \mathscr{A})$ be a measurable space. Assume that $f_n, f \in \mathscr{F}_{(X,\mathscr{A})}$ and $\alpha, \alpha_n \in [0, 1]$ with $f_n \searrow f$ and $\alpha_n \nearrow \alpha$. Then we have the following assertions:*

1. *$\{(f_n)_{\alpha_n}\}$, $\left\{(f_n)_{\alpha_n^+}\right\}$ are nonincreasing sequences and $f_\alpha \subset (f_n)_{\alpha_n}$, $f_{\alpha^+} \subset (f_n)_{\alpha_n^+}$ for all $n \in \mathbb{N}$.*
2. *$\bigcap\limits_{n=1}^{\infty} (f_n)_{\alpha_n} = f_\alpha$ and $f_{\alpha^+} \subset \bigcap_{n=1}^{\infty} (f_n)_{\alpha_n^+}$.*
3. *If $\{f_n\}$ is strictly decreasing or $\{\alpha_n\}$ is strictly increasing then $\bigcap\limits_{n=1}^{\infty} (f_n)_{\alpha_n} = \bigcap\limits_{n=1}^{\infty} (f_n)_{\alpha_n^+}$.*

Proof. 1) It is straightforward.
2) It is obvious that

$$f_\alpha \subset \bigcap_{n=1}^{\infty} (f_n)_{\alpha_n} \text{ and } f_{\alpha^+} \subset \bigcap_{n=1}^{\infty} (f_n)_{\alpha_n^+}.$$

Further, suppose that $x \in \bigcap_{n=1}^{\infty} (f_n)_{\alpha_n}$ then $f_n(x) \geqslant \alpha_n$ for all $n \in \mathbb{N}$. Passing limit $n \to \infty$ we get $f(x) \geqslant \alpha$. This implies that $x \in f_\alpha$ and the proof is finished.
3) It is sufficient to prove that

$$\bigcap_{n=1}^{\infty} (f_n)_{\alpha_n} \subset \bigcap_{n=1}^{\infty} (f_n)_{\alpha_n^+}.$$

We argue by contradiction i.e., suppose that $x \notin \bigcap_{n=1}^{\infty} (f_n)_{\alpha_n^+}$. Then there exists $n_0 \in \mathbb{N}$ such that

$$x \notin \left(f_{n_0}\right)_{\alpha_{n_0}^+} \text{ i.e., } f_{n_0}(x) \leqslant \alpha_{n_0}.$$

It deduces that

$$f_{n_0+1}(x) \leqslant f_{n_0}(x) \leqslant \alpha_{n_0} \leqslant \alpha_{n_0+1}.$$

From the strict decreasingness of $\{f_n\}$ or the strict increasingness of $\{\alpha_n\}$, it follows that

$$f_{n_0+1}(x) < \alpha_{n_0+1} \text{ i.e., } x \notin \left(f_{n_0+1}\right)_{\alpha_{n_0+1}}.$$

This means that

$$x \notin \bigcap_{n=1}^{\infty} (f_n)_{\alpha_n}.$$

The proof is finished. □

Lemma 3.2. *Let $(X, \mathscr{A})$ be a measurable space. Assume that $f_n, f \in \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ and $\alpha_n, \alpha \in [0,1]$ with $f_n \nearrow f$ and $\alpha_n \searrow \alpha$. Then we have the following assertions:*

1. *$\{(f_n)_{\alpha_n}\}$ and $\{(f_n)_{\alpha_n^+}\}$ are nondecreasing and $(f_n)_{\alpha_n} \subset f_\alpha$, $(f_n)_{\alpha_n^+} \subset f_{\alpha^+}$ for all $n \in \mathbb{N}$.*
2. *$\bigcup_{n=1}^{\infty} (f_n)_{\alpha_n} \subset f_\alpha$ and $\bigcup_{n=1}^{\infty} (f_n)_{\alpha_n^+} = f_{\alpha^+}$.*
3. *If $\{f_n\}$ is strictly increasing or $\{\alpha_n\}$ is strictly decreasing. Then*

$$\bigcup_{n=1}^{\infty} (f_n)_{\alpha_n^+} = \bigcup_{n=1}^{\infty} (f_n)_{\alpha_n}$$

Proof. 1) It is straightforward.
2) It is easy to see that

$$\bigcup_{n=1}^{\infty} (f_n)_{\alpha_n} \subset f_\alpha \text{ and } \bigcup_{n=1}^{\infty} (f_n)_{\alpha_n^+} \subset f_{\alpha^+}.$$

Further, suppose that $x \notin \bigcup_{n=1}^{\infty} (f_n)_{\alpha_n^+}$ then $x \notin (f_n)_{\alpha_n^+}$ for all $n \in \mathbb{N}$. This implies that $f_n(x) \leqslant \alpha_n$ for all $n \in \mathbb{N}$. By passing limit, one has $f(x) \leqslant \alpha$. This means that $x \notin f_{\alpha^+}$. Therefore,

$$\bigcup_{n=1}^{\infty} (f_n)_{\alpha_n^+} \supset f_{\alpha^+}.$$

So,

$$\bigcup_{n=1}^{\infty} (f_n)_{\alpha_n^+} = f_{\alpha^+}.$$

3) Suppose that $x \in \bigcup_{n=1}^{\infty} (f_n)_{\alpha_n}$. Then there exists $n_0 \in \mathbb{N}$ such that $x \in \left(f_{n_0}\right)_{\alpha_{n_0}}$. Thus,

$$f_{n_0+1}(x) \geqslant f_{n_0}(x) \geqslant \alpha_{n_0} \geqslant \alpha_{n_0+1}.$$

From the strict increasingness of $\{f_n\}$ or the strict decreasingness of $\{\alpha_n\}$, it follows that

$$f_{n_0+1}(x)) > \alpha_{n_0+1} \text{ i.e., } x \in \left(f_{n_0+1}\right)_{\alpha_{n_0+1}^+}.$$

This implies that

$$x \in \bigcup_{n=1}^{\infty} (f_n)_{\alpha_n^+}.$$

Then the proof is finished. □

The first main result of the paper is given by

Theorem 3.1. *Let $\mathsf{S} \in \mathfrak{S}$ be left-continuous in the first variable. Then we have*

$$\mathbf{I}_{\mathsf{S}}(\mu, f) = \sup_{\alpha \in [0,1]} \mathsf{S}(\alpha, \mu(f_{\alpha^+})) \quad \text{for every } \mu \in \mathscr{M}^1_{(X,\mathscr{A})},\ f \in \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}.$$

Proof. Without loss of generality, we can assume that $\mathbf{I}_{\mathsf{S}}(\mu, f) > 0$. Then for every $\varepsilon > 0$ small enough, there exists $\alpha_\varepsilon \in (0, 1]$ such that

$$0 < \mathbf{I}_{\mathsf{S}}(f, \mu) - \varepsilon < \mathsf{S}\left(\alpha_\varepsilon, \mu(f_{\alpha_\varepsilon})\right) \leqslant \mathbf{I}_{\mathsf{S}}(\mu, f). \tag{2}$$

Now, we claim that there exists $\beta \in (0, 1)$ such that

$$\mathbf{I}_{\mathsf{S}}(\mu, f) - \varepsilon < \mathsf{S}\left(\beta, \mu\left(f_{\beta^+}\right)\right).$$

Indeed, we argue by contradiction. Assume that for any $\gamma \in (0,1)$ there holds

$$\mathsf{S}\left(\gamma, \mu\left(f_{\gamma^+}\right)\right) \leqslant \mathbf{I}_{\mathsf{S}}(\mu, f) - \varepsilon.$$

Taking $\gamma_n = \alpha_\varepsilon - \frac{1}{n}$ for every $n \in \mathbb{N}$ large enough. Then $\gamma_n \in (0, \alpha_\varepsilon)$ for all $n \in \mathbb{N}$ and $\gamma_n \nearrow \alpha_\varepsilon$. Therefore,

$$\mathsf{S}\left(\gamma_n, \mu\left(f_{\alpha_\varepsilon}\right)\right) \leqslant \mathsf{S}\left(\gamma_n, \mu\left(f_{\gamma_n^+}\right)\right) \leqslant \mathbf{I}_{\mathsf{S}}(\mu, f) - \varepsilon.$$

Passing limit $n \to \infty$ and using the assumption on S, we obtain that

$$\lim_{n\to\infty} \mathsf{S}\left(\gamma_n, \mu\left(f_{\alpha_\varepsilon}\right)\right) = \mathsf{S}\left(\alpha_\varepsilon, \mu\left(f_{\alpha_\varepsilon}\right)\right) \leqslant \mathbf{I}_{\mathsf{S}}(\mu, f) - \varepsilon$$

which is a contradiction. This shows that the claim holds. So,

$$\mathbf{I}_{\mathsf{S}}(\mu, f) = \sup_{\alpha\in[0,1]} \mathsf{S}(\alpha, \mu(f_{\alpha^+})).$$

The proof is finished. □

We have the following interesting result.

Theorem 3.2. *Let $\mu \in \mathscr{M}^1_{(X,\mathscr{A})}$ be continuous from above, $\mathsf{S} \in \mathfrak{S}$ be right-continuous in each variable and $f \in \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$. Then the function $g: [0,1] \to [0,1]$ defined by: $g(\alpha) = \mathsf{S}(\alpha, \mu(f_\alpha))$ is upper semicontinuous.*

Proof. The conclusion of Theorem 3.2 will be proved by applying Proposition 2.3-3. Indeed, consider any sequence $\{\alpha_n\} \subset [0,1]$ with $\lim_{n\to\infty} \alpha_n = \alpha$. It is known that there exists a subsequence $\{\alpha_{n_k^{(1)}}\}_k$ of sequence $\{\alpha_n\}_n$ such that

$$\limsup_{n\to\infty} g(\alpha_n) = \lim_{k\to\infty} g\left(\alpha_{n_k^{(1)}}\right). \tag{3}$$

On the other hand, from $\lim_{k\to\infty} \alpha_{n_k^{(1)}} = \alpha$, it follows that there exists a subsequence $\{\alpha_{n_k^{(2)}}\}_k$ of sequence $\{\alpha_{n_k^{(1)}}\}_k$ such that

$$\text{either } \alpha_{n_k^{(2)}} \leqslant \alpha \text{ for all } k \in \mathbb{N} \text{ and } \alpha_{n_k^{(2)}} \nearrow \alpha \text{ or } \alpha_{n_k^{(2)}} \geqslant \alpha \text{ for all } k \in \mathbb{N} \text{ and } \alpha_{n_k^{(2)}} \searrow \alpha.$$

1) If $\alpha_{n_k^{(2)}} \leqslant \alpha$ for all $k \in \mathbb{N}$ and $\alpha_{n_k^{(2)}} \nearrow \alpha$ then $f_{\alpha_{n_k^{(2)}}} \searrow f_\alpha$. By applying the continuity from above of measure μ and Lemma 3.1-2, we get

$$\lim_{k\to\infty} \mu\left(f_{\alpha_{n_k^{(2)}}}\right) = \mu\left(\bigcap_{k=1}^{\infty} f_{\alpha_{n_k^{(2)}}}\right) = \mu(f_\alpha).$$

From the right-continuity in the second variable of semicopula S, it follows that

$$g\left(\alpha_{n_k^{(2)}}\right) = \mathsf{S}\left(\alpha_{n_k^{(2)}}, \mu(f_{\alpha_{n_k^{(2)}}})\right) \leqslant \mathsf{S}\left(\alpha, \mu(f_{\alpha_{n_k^{(2)}}})\right) \to \mathsf{S}(\alpha, \mu(f_\alpha)) \text{ as } k \to \infty. \tag{4}$$

From (3) and (4), we deduce that

$$\limsup_{n\to\infty} g(\alpha_n) \leqslant g(\alpha),$$

In view of Proposition 2.3-3, we complete the proof.
2) If $\alpha_{n_k^{(2)}} \geqslant \alpha$ for all $k \in \mathbb{N}$ and $\alpha_{n_k^{(2)}} \searrow \alpha$ then $f_{\alpha_{n_k^{(2)}}} \subset f_\alpha$ for all $k \in \mathbb{N}$. By applying the right-continuity in the first variable of S, it follows that

$$\begin{aligned}\lim_{k\to\infty} g\left(\alpha_{n_k^{(2)}}\right) &= \lim_{k\to\infty} \mathsf{S}\left(\alpha_{n_k^{(2)}}, \mu\left(f_{\alpha_{n_k^{(2)}}}\right)\right)\\ &\leqslant \lim_{k\to\infty} \mathsf{S}\left(\alpha_{n_k^{(2)}}, \mu\left(f_\alpha\right)\right)\\ &= \mathsf{S}\left(\lim_{k\to\infty} \alpha_{n_k^{(2)}}, \mu\left(f_\alpha\right)\right)\\ &= \mathsf{S}\left(\alpha, \mu\left(f_\alpha\right)\right)\\ &= g(\alpha).\end{aligned} \tag{5}$$

By combining (3) and (5), we obtain that

$$\limsup_{k\to\infty} g(\alpha_n) \leqslant g(\alpha).$$

So in all cases, the proof is finished by applying Proposition 2.3-3. □

Next, the second main result is given by

Theorem 3.3. *Let $\mu \in \mathscr{M}^1_{(X,\mathscr{A})}$ be continuous from above, $\mathsf{S} \in \mathfrak{S}$ be right-continuous in each variable and $f \in \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$. Then the function $\mathsf{S}(\alpha, \mu(f_\alpha))$ achieves its maximum on $[0, 1]$ i.e., there exists $\beta \in [0, 1]$ such that*

$$\mathsf{S}\left(\beta, \mu(f_\beta)\right) = \sup_{\alpha\in[0,1]} \mathsf{S}\left(\alpha, \mu(f_\alpha)\right) = \mathbf{I}_{\mathsf{S}}\left(\mu, f\right).$$

Proof. By applying Theorem 3.2 we get that the function $\alpha \mapsto \mathsf{S}(\alpha, \mu(f_\alpha))$ is upper semicontinuous. In view of Proposition 2.4, we finish the proof. □

4. An application

In this section, the main results of the paper are applied to formulate alternative proofs of Theorems 2.1 and 2.2.

4.1. Solving Theorem 2.1 (the implication $1 \Rightarrow 2$)

Proof. By using Theorem 3.1, one has

$$\begin{aligned}\lim_{n\to\infty} \mathbf{I}_{\mathsf{S}}(\mu, f_n) &= \lim_{n\to\infty} \sup_{\alpha\in[0,1]} \mathsf{S}\left(\alpha, \mu\left((f_n)_{\alpha^+}\right)\right)\\ &= \sup_{n\in\mathbb{N}} \sup_{\alpha\in[0,1]} \mathsf{S}\left(\alpha, \mu\left((f_n)_{\alpha^+}\right)\right)\\ &= \sup_{\alpha\in[0,1]} \sup_{n\in\mathbb{N}} \mathsf{S}\left(\alpha, \mu\left((f_n)_{\alpha^+}\right)\right)\\ &= \sup_{\alpha\in[0,1]} \lim_{n\to\infty} \mathsf{S}\left(\alpha, \mu\left((f_n)_{\alpha^+}\right)\right)\\ &= \sup_{\alpha\in[0,1]} \mathsf{S}\left(\alpha, \lim_{n\to\infty} \mu\left((f_n)_{\alpha^+}\right)\right) \text{ (By applying the left-continuity of } \mathsf{S} \text{ in the second variable)}\\ &= \sup_{\alpha\in[0,1]} \mathsf{S}\left(\alpha, \mu\left(\bigcup_{n\in\mathbb{N}} (f_n)_{\alpha^+}\right)\right) \text{ (By applying the continuity from below of } \mu)\\ &= \sup_{\alpha\in[0,1]} \mathsf{S}\left(\alpha, \mu\left(f_{\alpha^+}\right)\right) \text{ (By using Lemma 3.2-2)}\\ &= \mathbf{I}_{\mathsf{S}}(\mu, f) \text{ (By using Theorem 3.1 again).}\end{aligned}$$

The proof of the implication $1 \Rightarrow 2$ of Theorem 2.1 is finished. □

4.2. Solving Theorem 2.2 (the implication $1 \Rightarrow 2$)

Proof. From Theorem 3.3, it follows that for every $n \in \mathbb{N}$ there exists $\alpha_n \in [0, 1]$ such that

$$\mathsf{S}\left(\alpha_n, \mu\left(f_n\right)_{\alpha_n}\right) = \mathbf{I}_{\mathsf{S}}\left(\mu, f_n\right).$$

By applying Weierstrass theorem, there exists a subsequence $\{\alpha_{n_k^{(1)}}\}_k$ of sequence $\{\alpha_n\}_n$ such that $\alpha_{n_k^{(1)}} \to \alpha \in [0, 1]$. From this result it follows that there exists a subsequence $\{\alpha_{n_k^{(2)}}\}_k$ of sequence $\{\alpha_{n_k^{(1)}}\}_k$ such that

$$\text{either } \alpha_{n_k^{(2)}} \leqslant \alpha \text{ for all } k \in \mathbb{N} \text{ and } \alpha_{n_k^{(2)}} \nearrow \alpha \quad \text{or} \quad \alpha_{n_k^{(2)}} \geqslant \alpha \text{ for all } k \in \mathbb{N} \text{ and } \alpha_{n_k^{(2)}} \searrow \alpha.$$

1) If $\alpha_{n_k^{(2)}} \leqslant \alpha$ for all $k \in \mathbb{N}$ and $\alpha_{n_k^{(2)}} \nearrow \alpha$ then

$$\begin{aligned}
\lim_{n\to\infty} \mathbf{I}_{\mathsf{S}}\left(\mu, f_n\right) &= \lim_{k\to\infty} \mathsf{S}\left(\alpha_{n_k^{(2)}}, \mu\left(\left(f_{n_k^{(2)}}\right)_{\alpha_{n_k^{(2)}}}\right)\right) \\
&\leqslant \lim_{k\to\infty} \mathsf{S}\left(\alpha, \mu\left(\left(f_{n_k^{(2)}}\right)_{\alpha_{n_k^{(2)}}}\right)\right) \\
&= \mathsf{S}\left(\alpha, \lim_{k\to\infty} \mu\left(\left(f_{n_k^{(2)}}\right)_{\alpha_{n_k^{(2)}}}\right)\right) \text{ (By applying the right-continuity of } \mathsf{S}) \\
&= \mathsf{S}\left(\alpha, \mu\left(\bigcap_{k=1}^{\infty}\left(f_{n_k^{(2)}}\right)_{\alpha_{n_k^{(2)}}}\right)\right) \text{ (By applying the continuity from above of } \mu) \\
&= \mathsf{S}\left(\alpha, \mu\left(f_\alpha\right)\right) \text{ (By using Lemma 3.1-2).}
\end{aligned}$$

This implies that

$$\lim_{n\to\infty} \mathbf{I}_{\mathsf{S}}\left(\mu, f_n\right) = \mathbf{I}_{\mathsf{S}}\left(\mu, f\right).$$

2) If $\alpha_{n_k^{(2)}} \geqslant \alpha$ for all $k \in \mathbb{N}$ and $\alpha_{n_k^{(2)}} \searrow \alpha$ then

$$\begin{aligned}
\lim_{n\to\infty} \mathbf{I}_{\mathsf{S}}\left(\mu, f_n\right) &= \lim_{k\to\infty} \mathsf{S}\left(\alpha_{n_k^{(2)}}, \mu\left(\left(f_{n_k^{(2)}}\right)_{\alpha_{n_k^{(2)}}}\right)\right) \\
&\leqslant \lim_{k\to\infty} \mathsf{S}\left(\alpha_{n_k^{(2)}}, \mu\left(\left(f_{n_k^{(2)}}\right)_{\alpha}\right)\right) \\
&= \mathsf{S}\left(\lim_{k\to\infty} \alpha_{n_k^{(2)}}, \lim_{k\to\infty} \mu\left(\left(f_{n_k^{(2)}}\right)_{\alpha}\right)\right) \text{ (By the right-continuity of } \mathsf{S}) \\
&= \mathsf{S}\left(\alpha, \mu\left(\bigcap_{k=1}^{\infty}\left(f_{n_k^{(2)}}\right)_{\alpha}\right)\right) \text{ (By the continuity from above of } \mu) \\
&= \mathsf{S}\left(\alpha, \mu\left(f_\alpha\right)\right) \text{ (By using Lemma 3.1-2).}
\end{aligned}$$

This shows that

$$\lim_{n\to\infty} \mathbf{I}_{\mathsf{S}}\left(\mu, f_n\right) = \mathbf{I}_{\mathsf{S}}\left(\mu, f\right).$$

So, the proof of the implication $1 \Rightarrow 2$ of Theorem 2.2 is completed. □

5. Some versions of monotone convergence theorems

In this section, we show some other versions of monotone convergence theorems.
Recall that $\mu \in \mathscr{M}^1_{(X,\mathscr{A})}$ is called to be null-additive if $\mu(A \cup B) = \mu(A)$ for all $A, B \in \mathscr{A}$ with $\mu(B) = 0$. The following result shows that the assumption of semicopula S in Theorem 2.1 can be mitigated.

Theorem 5.1. *Let $\mu \in \mathscr{M}^1_{(X,\mathscr{A})}$ be continuous from below, null-additive with $\mu(\{a\}) = 0$ for all $a \in X$. Let $\mathsf{S} \in \mathfrak{S}$ be left-continuous in the second variable. Assume that $f, f_n \in \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ with $f_n \nearrow f$, f be injective and $\mathbf{I}_{\mathsf{S}}(\mu, f) = \sup_{\alpha \in (0,1)} \mathsf{S}(\alpha, \mu(f_\alpha))$. Then we get*

$$\lim_{n\to\infty} \mathbf{I}_{\mathsf{S}}(\mu, f_n) = \mathbf{I}_{\mathsf{S}}(\mu, f).$$

Proof. The proof is given in Appendix. □

The following result is a modification of Theorem 2.1.

Theorem 5.2. *Let $\mathsf{S} \in \mathfrak{S}$ be left-continuous, $\mu \in \mathscr{M}^1_{(X,\mathscr{A})}$ be continuous from below and null-additive. Then for all $f, f_n \in \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ with $f_n \nearrow$ and $f_n \xrightarrow{\mu-a.u.} f$, it holds*

$$\lim_{n\to\infty} \mathbf{I}_{\mathsf{S}}(\mu, f_n) = \mathbf{I}_{\mathsf{S}}(\mu, f).$$

Proof. The proof is given in Appendix. □

Next, a modification of Theorem 2.2 is given by:

Theorem 5.3. *Let $\mathsf{S} \in \mathfrak{S}$ be right-continuous, $\mu \in \mathscr{M}^1_{(X,\mathscr{A})}$ be continuous from above and null-additive. Then for all $f, f_n \in \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ with $f_n \searrow$ and $f_n \xrightarrow{\mu-a.u.} f$, it holds*

$$\lim_{n\to\infty} \mathbf{I}_{\mathsf{S}}(\mu, f_n) = \mathbf{I}_{\mathsf{S}}(\mu, f).$$

Proof. It is the same as the proof of Theorem 5.2. So, it is omitted. □

Remark 5.1. Theorems 5.2 and 5.3 are also modified versions of Theorems 2.8 and 2.7 in [6].

6. Conclusion

The two main results in our paper are
1) Let $\mathsf{S} \in \mathfrak{S}$ be left-continuous in the first variable. Then

$$\mathbf{I}_{\mathsf{S}}(\mu, f) = \sup_{\alpha \in [0,1]} \mathsf{S}(\alpha, \mu(f_{\alpha^+})) \quad \text{for every } \mu \in \mathscr{M}^1_{(X,\mathscr{A})},\ f \in \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}.$$

2) Let $\mathsf{S} \in \mathfrak{S}$ be right-continuous and $\mu \in \mathscr{M}^1_{(X,\mathscr{A})}$ be continuous from above. Then for every $f \in \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ there exists $\beta \in [0, 1]$ such that

$$\mathbf{I}_{\mathsf{S}}(\mu, f) = \mathsf{S}\left(\beta, \mu(f_\beta)\right).$$

By applying the above results, we provided alternative proofs of monotone convergence theorems for smallest semicopula-based universal integrals, which are proposed by J. Borzová-Molnárová et al. in [6].
Through our work in the paper, we would like to propose the following open problems:

Problem 1. Characterize all the semicopulas S for which the property

$$\mathbf{I}_{\mathsf{S}}(\mu, f) = \sup_{\alpha\in[0,1]} \mathsf{S}(\alpha, \mu(f_{\alpha^+}))$$

holds for all $(X, \mathscr{A}) \in \mathscr{S}$, $\mu \in \mathscr{M}^1_{(X,\mathscr{A})}$ and $f \in \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$.

Problem 2. Let $(X, \mathscr{A}) \in \mathscr{S}$. Characterize all the semicopulas S and $\mu \in \mathscr{M}^1_{(X,\mathscr{A})}$ such that for every $f \in \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ there exists $\beta \in [0, 1]$ satisfying

$$\mathbf{I}_{\mathsf{S}}(\mu, f) = \mathsf{S}\left(\beta, \mu(f_\beta)\right).$$

In the development of this research direction, we expect that our work will be one of the useful references for researchers.

7. Appendix

This section is devoted to prove Proposition 2.2, Theorems 5.1 and 5.2.

The proof of Proposition 2.2.

Proof. i) S is left-continuous $\Leftrightarrow$ S is left-continuous in each variable. Indeed, assume that S is left-continuous in each variable. For any $a, b \in [0, 1]$, $x_n \to a^-$ and $b_n \to b^-$. We claim that

$$\mathsf{S}(x_n, y_n) \to \mathsf{S}(a, b).$$

To archive the claim we first put $a_n = \inf_{k\geqslant n} x_k$ and $b_n = \inf_{k\geqslant n} y_k$. Then $a_n \leqslant x_n$, $b_n \leqslant y_n$ and $a_n \nearrow a, b_n \nearrow b$. This implies that

$$\mathsf{S}(a_n, b_n) \nearrow \text{ and } \mathsf{S}(a_n, b_n) \leqslant \mathsf{S}(x_n, y_n) \leqslant \mathsf{S}(a, b). \tag{6}$$

Again put: $\alpha = \lim_{n\to\infty} \mathsf{S}(a_n, b_n) = \sup_{n\in\mathbb{N}} \mathsf{S}(a_n, b_n)$. On the other hand, we see that

$$\mathsf{S}(a_n, b_m) \leqslant \mathsf{S}(a_n, b_n) \leqslant \mathsf{S}(a, b) \text{ for all } m, n \in \mathbb{N} \text{ with } m \leqslant n.$$

Passing limit $n \to \infty$, we obtain that

$$\mathsf{S}(a, b_m) \leqslant \alpha \leqslant \mathsf{S}(a, b) \text{ for all } m \in \mathbb{N}.$$

Passing limit $m \to \infty$, we get that

$$\alpha = \mathsf{S}(a, b).$$

Combining this result with the estimate (6), we have

$$\mathsf{S}(x_n, y_n) \to \mathsf{S}(a, b).$$

So, S is left-continuous.
ii) S is right-continuous $\Leftrightarrow$ S is right-continuous in each variable. Indeed, assume that S is right-continuous in each variable. For any $a, b \in [0, 1]$, $x_n \to a^+$ and $b_n \to b^+$. We claim that

$$\mathsf{S}(x_n, y_n) \to \mathsf{S}(a, b).$$

To archive this claim we first put $a_n = \sup_{k\geqslant n} x_k$ and $b_n = \sup_{k\geqslant n} y_k$. Then $a_n \geqslant x_n$, $b_n \geqslant y_n$ and $a_n \searrow a, b_n \searrow b$. This implies that

$$\mathsf{S}(a_n, b_n) \searrow \text{ and } \mathsf{S}(a, b) \leqslant \mathsf{S}(x_n, y_n) \leqslant \mathsf{S}(a_n, b_n). \tag{7}$$

Again put: $\beta = \lim_{n\to\infty} \mathsf{S}(a_n, b_n) = \inf_{n\in\mathbb{N}} \mathsf{S}(a_n, b_n)$. On the other hand, we see that

$$\mathsf{S}(a,b) \leqslant \mathsf{S}(a_n, b_n) \leqslant \mathsf{S}(a_n, b_m) \text{ for all } m, n \in \mathbb{N} \text{ with } m \leqslant n.$$

Passing limit $n \to \infty$, we obtain that

$$\mathsf{S}(a,b) \leqslant \beta \leqslant \mathsf{S}(a, b_m) \text{ for all } m \in \mathbb{N}.$$

Passing limit $m \to \infty$, we get that

$$\beta = \mathsf{S}(a,b).$$

Combining this result with the estimate (7), we have

$$\mathsf{S}(x_n, y_n) \to \mathsf{S}(a,b).$$

So, S is right-continuous. □

The proof of Theorem 5.1.

Proof. It is easy to see that $f_\alpha = f_{\alpha^+} \cup f^{-1}(\alpha)$ for every $\alpha \in [0,1]$. On the other hand, from the assumptions it follows that $\mu\left(f^{-1}(\alpha)\right) = 0$. Therefore, $\mu(f_\alpha) = \mu(f_{\alpha^+})$ for all $\alpha \in [0,1]$. Furthermore,

$$\begin{aligned}
\lim_{n\to\infty} \mathbf{I}_{\mathsf{S}}(\mu, f_n) &= \lim_{n\to\infty} \sup_{\alpha\in[0,1]} \mathsf{S}\left(\alpha, \mu\left((f_n)_\alpha\right)\right) \\
&\geqslant \lim_{n\to\infty} \sup_{\alpha\in(0,1)} \mathsf{S}\left(\alpha, \mu\left((f_n)_{\alpha^+}\right)\right) \\
&= \sup_{n\in\mathbb{N}} \sup_{\alpha\in(0,1)} \mathsf{S}\left(\alpha, \mu\left((f_n)_{\alpha^+}\right)\right) \\
&= \sup_{\alpha\in(0,1)} \sup_{n\in\mathbb{N}} \mathsf{S}\left(\alpha, \mu\left((f_n)_{\alpha^+}\right)\right) \\
&= \sup_{\alpha\in(0,1)} \lim_{n\to\infty} \mathsf{S}\left(\alpha, \mu\left((f_n)_{\alpha^+}\right)\right) \\
&= \sup_{\alpha\in(0,1)} \lim_{n\to\infty} \mathsf{S}\left(\alpha, \mu\left((f_n)_{\alpha^+}\right)\right) \\
&= \sup_{\alpha\in(0,1)} \mathsf{S}\left(\alpha, \bigcup_{n\in\mathbb{N}} \mu\left((f_n)_{\alpha^+}\right)\right) \\
&= \sup_{\alpha\in(0,1)} \mathsf{S}\left(\alpha, \mu(f_{\alpha^+})\right) \\
&= \mathbf{I}_{\mathsf{S}}(\mu, f).
\end{aligned}$$

This implies that

$$\lim_{n\to\infty} \mathbf{I}_{\mathsf{S}}(\mu, f_n) = \mathbf{I}_{\mathsf{S}}(\mu, f).$$

The proof is finished. □

The proof of Theorem 5.2.

Proof. From the assumption that $f_n \xrightarrow{\mu-a.u.} f$, we deduce that $f_n \xrightarrow{\mu-a.e.} f$ i.e., there exists $A \in \mathscr{A}$ with $\mu(A) = 0$ such that $f_n \cdot \chi_A \nearrow f \cdot \chi_A$. By applying Theorem 2.1, we obtain that

$$\lim_{n\to\infty} \mathbf{I}_{\mathsf{S}}(\mu, f_n \cdot \chi_A) = \mathbf{I}_{\mathsf{S}}(\mu, f \cdot \chi_A). \tag{8}$$

On the other hand, from the null-additivity of μ it follows that

$$\mathsf{S}(\mu, f_n \cdot \chi_A) = \mathsf{S}(\mu, f_n) \text{ and } \mathsf{S}(\mu, f \cdot \chi_A) = \mathsf{S}(\mu, f). \tag{9}$$

By combining (8) and (9), we finish the proof. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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